Relativistic kinematics

4-momentum for a particle of mass m:

$p=(E/c, p_x, p_y, p_z)$ where

total energy: $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v} = \gamma m\beta c$

The line element is an invariant $$ds^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

If 4-vectors transform like ds, the scalar product of themselves is invariant under Lorentz transformations: $P = \{P_t,P_x,P_y,P_z\}$

$P^2 = -P_x^2-P_y^2-P_z^2+P_t^2$ or for 2 4-vectors $PQ= -P_xQ_x-P_yQ_y-P_zQ_z+P_tQ_t$

For the energy-momentum 4-vector:

In the rest frame $E = mc^2 \Rightarrow P = (mc,\mathbf{0}) \Rightarrow P^2 = m^2c^2$. This is the same value it has in any ref system:

$P^2 = (E/c)^2-\mathbf{p}^2 = \gamma^2m^2c^2 - \gamma^2m^2v^2 = \gamma^2m^2c^2 (1- v^2/c^2) = m^2c^2$

Hence the total energy is

$$E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$$

Lorentz transformation of energy-momentum

Given a particle of energy E and momentum **p, the 4-momentum is** P=(E,p)

And $P^2 = m^2$ (c=1)

The velocity of the particle is β = **p**/E

The energy and momentum viewed from the frame moving with velocity β_f is

$$\begin{pmatrix} E^* \\ p^*_{\parallel} \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f \beta_f \\ -\gamma_f \beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_{\parallel} \end{pmatrix}, \qquad p^*_T = p_T$$

$p_{\parallel}$ is the component parallel to β_f

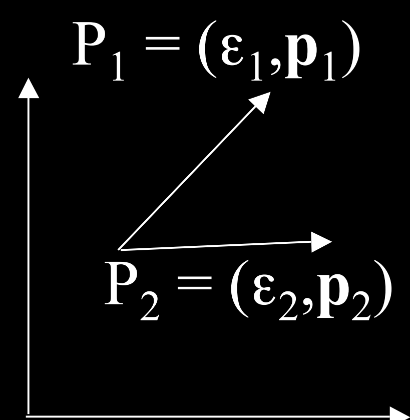
And $p_{\perp}$ is the orthogonal one

CM and laboratory systems

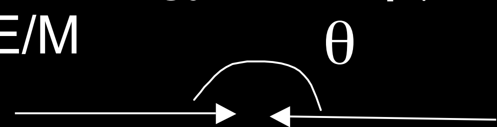
We have in a certain frame (laboratory) 2 particles with 4-momenta P_1 and P_2
What is the CM energy?

Let's consider 3 invariants
$P_1^2=m_1^2$ $P_2^2=m_2^2$ $[P_1P_2$ or $(P_1 \pm P_2)^2]$

In CM:
$\mathbf{p}^*_1+\mathbf{p}^*_2 = 0$ Hence $P^*_1+P^*_2 = (\varepsilon^*_1+\varepsilon^*_2, \mathbf{0}) = (E^*, \mathbf{0})$
$E^{*2} = (\varepsilon^*_1+\varepsilon^*_2)^2 = (P^*_1+P^*_2)^2 = (P_1+P_2)^2$
If M and P are the total mass and energy-momentum
$P^2 = (P_1+P_2)^2 = M^2 = E^{*2}$ and since it is an invariant
$P^2 = (\varepsilon_1+\varepsilon_2)^2 - (\mathbf{p}_1+\mathbf{p}_2)^2$
Any 4-vector can be written as $\mathbf{p} = m\mathbf{v}\gamma$ and $\varepsilon = m\gamma$
So for the tot momentum and energy $\mathbf{P} = M\beta\gamma$ and $E = M\gamma$ (we assume $c=1$) $\Rightarrow \beta_{CM} = \mathbf{p}/E$ and $\gamma_{CM} = E/M$



Laboratory system



Collisions of 2 particles m_1 and m_2 at an angle of θ one respect to the other:
$E^* = P = [(\varepsilon_1+\varepsilon_2)^2-(\mathbf{p}_1+\mathbf{p}_2)^2]^{1/2} = [m_1^2+m_2^2+2\varepsilon_1\varepsilon_2(1-\beta_1\beta_2\cos\theta)]^{1/2}$

In a e^+e^- collider $E_1 = E_2$, $m_1 = m_2 = m \ll E$, $\beta_1=\beta_2\approx 1$ $\theta = 180^\circ \Rightarrow E^* \sim 2E$
In a fixed target experiment: $m_2 = M \gg m_1$ and $\beta_2 = 0$, $E_2 = M \Rightarrow E^* \sim \sqrt{(2EM)}$ $E \gg M$

Examples

Problem: suppose we move with particle 1, which is for us the energy of particle 2?

In the rest frame of 1 (laboratory frame): $\mathbf{p}_1 = 0$ and $\varepsilon_1 = m_1$

$\Rightarrow P_1P_2$ = invariant = $\varepsilon_1\varepsilon_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = m_1\varepsilon_2$

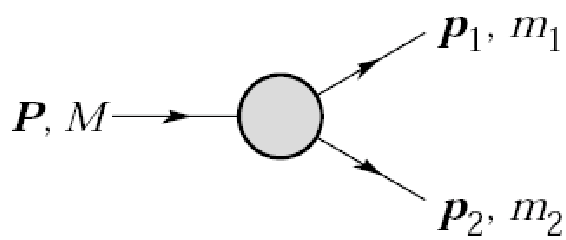
So the energy of particle 2 in reference 1 is:

$E_{21} = \varepsilon_2 = P_1P_2/m_1$ that is also an invariant

$$|\mathbf{p}_{21}|^2 = E_{21}^2 - m_2^2 = \frac{(P_1P_2)^2 - m_1^2 m_2^2}{m_1^2}$$

These expressions are invariant and can be evaluated in any ref system

Examples



- 2-body decay

Conservation of energy and momentum:

$$M = E_1 + E_2 = \sqrt{|\mathbf{p}_1|^2 + m_1^2} + \sqrt{|\mathbf{p}_2|^2 + m_2^2} \Rightarrow M^2 + |\mathbf{p}_1|^2 + m_1^2 - 2M\sqrt{|\mathbf{p}_1|^2 + m_1^2} = |\mathbf{p}_2|^2 + m_2^2$$

$$|\mathbf{p}_1| = -|\mathbf{p}_2|$$

$$E_1 = \frac{M^2 + \left(m_1^2 - m_2^2\right)}{2M}$$

$$E_2 = \frac{M^2 - \left(m_1^2 - m_2^2\right)}{2M} \qquad |\mathbf{p}_1|^2 = -|\mathbf{p}_2|^2 = \frac{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}{4M^2}$$

$$|\mathbf{p}_1| = -|\mathbf{p}_2| = \frac{\sqrt{[M^2 - (m_1^2 + m_2^2)][M^2 - (m_1^2 - m_2^2)]}}{2M}$$

Eg

$\pi \rightarrow \mu + \nu$ and $m_\nu = 0$ ($E_\nu = |\mathbf{p}_\nu|$)

$$m_\pi = E_\nu + E_\mu = |\mathbf{p}_\nu| + \sqrt{|\mathbf{p}_\mu|^2 + m_\mu^2} \Rightarrow m_\pi^2 + E_\nu^2 - 2m_\pi E_\nu = |\mathbf{p}_\mu|^2 + m_\mu^2 \Rightarrow E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = \frac{m_\pi}{4}$$

$$\mathbf{p}_\nu = -\mathbf{p}_\mu$$

Reaction thresholds

Energy of projectile to produce particles in the final state at rest

m_p, p_p → ← m_t, p_t $s = E_{cm}^2$ $c=1$

True in any reference system

$$E_{tot} = E_{kin} + m = \sqrt{|\mathbf{p}|^2 + m^2} \Rightarrow E_{kin}^2 + m^2 + 2E_{kin}m = m^2 + |\mathbf{p}|^2 \Rightarrow E_{kin}^2 = |\mathbf{p}|^2 - 2E_{kin}m$$

$$t + p \to M_1 + M_2 + \ldots M_n$$

$$\sqrt{s} = \sum_f M_f = \sqrt{E_{tot}^2 - |\mathbf{p_{tot}}|^2}$$

$$E_{tot} = m_p + m_t + E_{kin,p}$$

$$|\mathbf{p_{tot}}| = |\mathbf{p_p}|$$

In the ref frame where the target is at rest

$$\left(\sum_f M_f\right)^2 = (m_t + m_p + E_{kin,p})^2 - |\mathbf{p_p}|^2$$

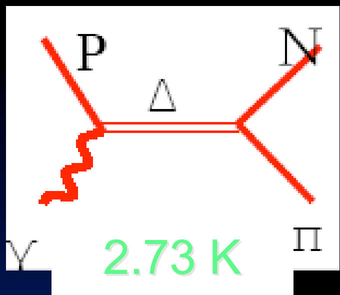
$$\left(\sum_f M_f\right)^2 = (m_t + m_p)^2 + E_{kin,p}{}^2 + 2(m_t + m_p)E_{kin,p} - |\mathbf{p_p}|^2$$

$$\left(\sum_f M_f\right)^2 = (m_t + m_p)^2 + E_{kin,p}{}^2 + 2(m_t + m_p)E_{kin,p} - |\mathbf{p_p}|^2 = (m_t + m_p)^2 + 2m_t E_{kin,p}$$

$$E_{kin,p} = \frac{\left(\sum_f M_f\right)^2 - (m_t + m_p)^2}{2m_t}$$

Threshold for GZK cut-off

[Greisen 66; Zatsepin & Kuzmin66]



Threshold for p-γ→Δ→πN

$$E_{k\gamma} = \frac{\left(\sum_f M_f\right)^2 - (m_t + m_p)^2}{2m_t} = \frac{(m_N + m_\pi)^2 - m_p^2}{2m_p}$$

$$\gamma + p \rightarrow \Delta^+ \rightarrow p\pi^0$$

$$\gamma + p \rightarrow \Delta^+ \rightarrow n\pi^+$$

$$E_\gamma = 145\text{MeV}$$

$$E_\gamma = 150\text{MeV}$$

in frame where p is at rest

Energy of CMB photons: =3k_BT effective energy for Planck spectrum

$\varepsilon_\gamma = 3\ [2.73]\ 8.62\ 10^{-5}$ eV

And their energy in the laboratory frame (rest frame of proton) is

$$E_\gamma \sim \gamma_p \varepsilon_\gamma = 150\text{MeV}$$

⇒γ_p= 2· 10^{11} and the threshold energy of the proton is then $E_p \sim \gamma_p m_p = 2 \cdot 10^{20}$ eV

Integrating over Planck spectrum $E_{p,th} \sim 5 \cdot 10^{19}$ eV

Transformations of velocity

$$t' = \gamma t - \gamma \frac{vx}{c^2}$$

$$x' = \gamma x - \gamma v t$$

If a point has velocity **u**' in the frame K' the velocity **u** in K is given by

$$x = \gamma(x' + \mathrm{v}t') \Rightarrow dx = \gamma(dx' + \mathrm{v}dt')$$

$$t = \gamma\left(t' + \frac{\mathrm{v}x'}{c^2}\right) \Rightarrow dt = \gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)$$

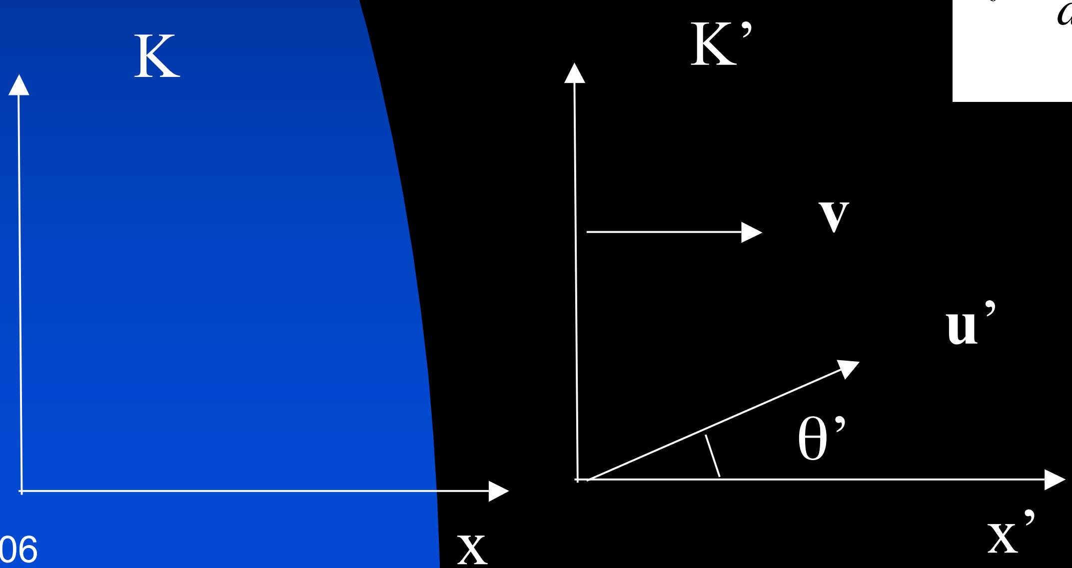
$$dy = dy'$$

$$dz = dz'$$

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + \mathrm{v}dt')}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_x + \mathrm{v}}{1 + \frac{\mathrm{v}u'_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{\mathrm{v}u'_x}{c^2}\right)}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_z}{\gamma\left(1 + \frac{\mathrm{v}u'_x}{c^2}\right)}$$



Transformations of velocity

Hence the generalization of these equations to an arbitrary velocity **v** not necessarily along x can be stated in terms of components of **u** parallel and perpendicular to **v**:

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + \mathrm{v}dt')}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_x + \mathrm{v}}{1 + \frac{\mathrm{v}u'_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{\mathrm{v}u'_x}{c^2}\right)}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{\mathrm{v}dx'}{c^2}\right)} = \frac{u'_z}{\gamma\left(1 + \frac{\mathrm{v}u'_x}{c^2}\right)}$$

$$u_{\parallel} = \frac{u'_{\parallel} + \mathrm{v}}{\left(1 + \frac{\mathrm{v}u'_{\parallel}}{c^2}\right)} \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + \frac{\mathrm{v}u'_{\parallel}}{c^2}\right)}$$

The directions of the velocities in the 2 frames are related by the aberration formula

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u'_{\perp}}{\gamma\left(1 + \frac{\mathrm{v}u'_{\parallel}}{c^2}\right)} \frac{\left(1 + \frac{\mathrm{v}u'_{\parallel}}{c^2}\right)}{u'_{\parallel} + \mathrm{v}} = \frac{\mathrm{u}'\sin\theta'}{\gamma(u'\cos\theta' + \mathrm{v})}$$

And the aberration of light is obtained for u’=c

$$\tan\theta = \frac{\mathrm{c}\sin\theta'}{\gamma(c\cos\theta' + \mathrm{v})} = \frac{\sin\theta'}{\gamma(\cos\theta' + \mathrm{v}/\mathrm{c})}$$

Aberration and beaming

Aberration is the apparent change in the direction of a moving object when the observer is also moving

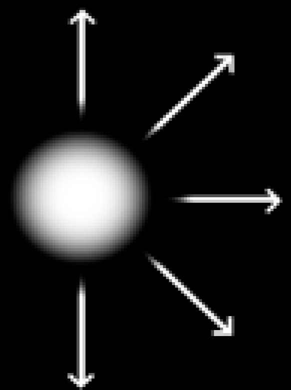
For $\theta' = \pi/2$

γ emitted perpendicular to **v** in K'

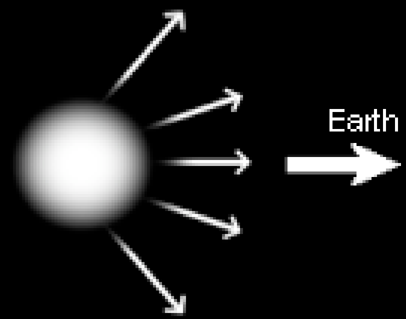
$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \mathrm{v}/c)} = \frac{c}{\gamma \mathrm{v}}$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\frac{c}{\gamma \mathrm{v}}}{\sqrt{1+\left(\frac{c}{\gamma \mathrm{v}}\right)^2}} = \frac{c}{\sqrt{\gamma^2 \mathrm{v}^2 + c^2}} = \frac{1}{\sqrt{\gamma^2\beta^2+1}} = \sqrt{\frac{1-\beta^2}{\beta^2+1-\beta^2}} = \sqrt{1-\beta^2} = \frac{1}{\gamma}$$

For highly relativistic speed $\gamma \gg 1$ and θ becomes small and $\theta \sim 1/\gamma$



Direction of emitted photons when at rest.



Direction of emitted photons when moving at speeds near c.

In K photons are concentrated in the forward direction. Very few photons are emitted with $\theta \gg 1/\gamma$

Aberration and beaming

If we are at rest in the spacecraft we see light coming from every direction from stars, but if the spacecraft travels at relativistic speeds the whole field of view Shrinks and even photons coming from behind, look as coming from the forward direction. If the ship travels towards Orion…