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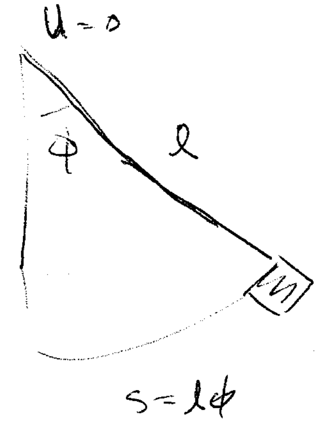
Ex: Pendulum



$$U = mgz = -mgl\cos\phi$$

$$F = -\frac{du}{ds} = -\frac{1}{l}\frac{du}{d\phi}$$

$$= -\frac{1}{l}(-mgl)(-\sin\phi)$$

$$= -mg\sin\phi$$

$$F = ma = m\ddot{s} = -ml\ddot{\phi}$$

$$\Rightarrow \quad \ddot{\phi} = -\frac{g}{l}\sin\phi$$

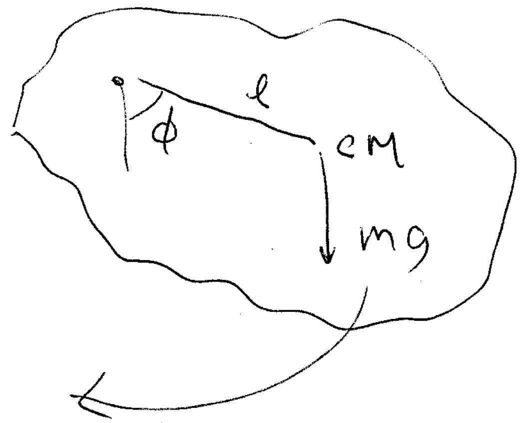
Is this SHO? No if ϕ is large

but everything is SHO near bottom of potential

For small ϕ, $\sin\phi \simeq \phi$

$$\Rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{l}}$$

Ex: Physical Pendulum



$$\tau = I\alpha$$

$$-\ell mg \sin\phi = I\ddot{\phi}$$

$$\ddot{\phi} = -\frac{mg\ell}{I}\sin\phi \simeq -\frac{mg\ell}{I}\phi$$

$$\Rightarrow \boxed{\omega^2 = \frac{mg\ell}{I}}$$

Demo

Damped Oscillation

$$ma = F = -\underbrace{m\omega_0^2}_{k} x - m\underset{\substack{\uparrow \\ \text{damping}}}{\gamma} v$$

$$\ddot{x} = -\omega_0^2 x - \gamma \frac{dx}{dt} \rightarrow$$ easier to solve using exponential

Note : $e^{i\theta} = \cos\theta + i\sin\theta$ [can show using Taylor series]

Guess : $x(t) \sim e^{-\gamma t} \cos\omega t$

$$\sim \frac{1}{2}\left(e^{(-\gamma + i\omega)t} + e^{(-\gamma - i\omega t)} \right)$$

Why exponentials are easy?

- differentiation is easy
- can describe both damping & oscillations in a unified way

Try $X = Ae^{i\alpha t}$ where α is an unknown <u>imaginary</u> number

$$(i\alpha)^2 x = -\omega_0^2 x - i\alpha\gamma x$$

$$\Rightarrow \quad \alpha^2 - i\gamma\alpha - \omega_0^2 = 0$$

$$\Rightarrow \quad \alpha = \frac{i\gamma \pm \sqrt{(-i\gamma)^2 - 4(-\omega_0^2)}}{2}$$

$$= \frac{i\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2} \equiv \alpha_\pm$$

Strong damping if $\omega_0 < \gamma \Rightarrow \alpha$ purely imaginary

$\Rightarrow$ no oscillations

Weak damping if $\omega_0 >> \gamma \Rightarrow$ oscillations

Solution: $x = A e^{i\alpha_\pm t}$

$$= \underbrace{A e^{-\frac{\gamma t}{2} \pm i \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} t}}_{\text{complex number}}$$

What this really means is the real part of RHS

$$x(t) = e^{-\frac{\gamma t}{2}} \operatorname{Re}\left(A e^{i \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} t}\right)$$