PHY 248 Lecture 6 Gary Shiu 2/2/07

Space Time warping

- In Special relativity

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- Gravitational redshift

$$dt^2 \to \left(1+\frac{\Phi}{c^2}\right)^2 dt^2$$

$$\simeq \left(1+\frac{2\Phi}{c^2}\right) dt^2$$ time is "warped"

- By general covariance (i.e. relativity unifies space & time)

$\Rightarrow$ space is warped as well

- In GR

$$ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)c^2 dt^2 + \frac{dr^2}{1+\frac{2\Phi}{c^2}} + r^2(d\theta^2+\sin^2\theta d\phi^2)$$

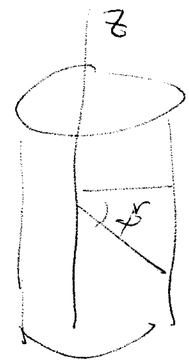
for problem with spherical symmetry

$\boxed{\text{Aside}}$ (can skip if you accept spacetime warping)

Consider the analogy of a rotating ref. frame discussed before

Convenient to work in cylindrical coordinates

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\phi^2 + \underbrace{dz^2}_{\text{redundant here}}$$



Let $\phi' = \phi - \omega t$ in rotating coordinate system

$$d\phi' = d\phi - \omega dt$$

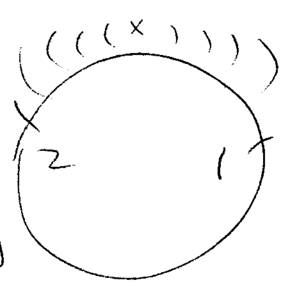
$$\Rightarrow \quad ds^2 = -c^2 dt^2 + dr^2 + r^2 (d\phi' + \omega\, dt)^2 + dz^2$$

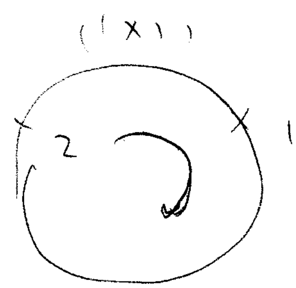
$$= -c^2 dt^2 + dr^2 + r^2 (d\phi'^2 + 2\omega d\phi' dt + \omega^2 dt^2) + dz^2$$

$$= -(c^2 - \omega^2 r^2) dt^2 + dr^2 + r^2 d\phi'^2 + \underbrace{2\omega r^2 d\phi' dt}_{\text{space also affected}} + dz^2$$

To appreciate what's going on, recall that clocks synchronized in original ref frame are not synchronized in rotating ref frame

clock 1 & 2 receives signal simultaneously





clock 2 receives signal first

$$dt' = \gamma\left(dt - \frac{v\,dx}{c^2}\right)$$

$$\simeq dt - \frac{\gamma v}{c^2}(\gamma r\,d\phi')$$

$$= dt - \frac{\gamma^2 v r\,d\phi'}{c^2} \qquad v = \omega r$$

$$= dt - \frac{\omega r^2\,d\phi'}{c^2 - \omega^2 r^2}$$

Hence

$$ds^2 = -(c^2-\omega^2 r^2)\left[dt' + \frac{\omega r^2\,d\phi'}{c^2-\omega^2 r^2}\right]^2$$

$$+ dr^2 + r^2\,d\phi'^2$$

$$+ 2\omega r^2\left[dt' + \frac{\omega r^2\,d\phi'}{c^2-\omega^2 r^2}\right]d\phi' + dz^2$$

$$= -(c^2-\omega^2 r^2)\,dt'^2 + dr^2 + \frac{c^2 r^2\,d\phi'^2}{c^2-\omega^2 r^2} + dz^2$$

Space time is warped.

$$\boxed{\omega^2 r^2 = 2 \times \frac{\frac{1}{2}L\omega^2}{m} = 2\Phi}$$

End of Aside

In our discussion of gravitational redshift, the gravitational accelaration is pointing towards z direction, analogously expect

$$ds^2 = -\left(1+\frac{2gh}{c^2}\right)c^2dt^2 + \frac{dz^2}{1+\frac{2gh}{c^2}} + dx^2 + dy^2.$$

For spacetime outside a spherical source

$$ds^2 = -\left(1+\frac{2\phi}{c^2}\right)c^2dt^2 + \frac{dr^2}{1+\frac{2\phi}{c^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

spherical coordinates

$$\phi(r) = -\frac{GM}{r}$$

This metric is called Schwarzchild metric which solves the equations of General relativity called Einstein eqns

General Relativity – Geometric Theory of Gravity

Spacetime is warped in the presence of gravity

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= g_{tt}\, dt^2 + 2g_{tx}\, dt\, dx + g_{xx}\, dx^2 + \ldots$$

$g_{\mu\nu}$ ≡ metric = fundamental physical quantities

As seen, $g_{tt} \simeq 1 + \frac{2\Phi}{c^2}$ ← gravitational potential

Newton's law $\Phi = -\frac{GM}{r}$

Einstein EQNs

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

is what replaces Newton's law of gravitation.

Many equations : many gravitational potentials $g_{\mu\nu}$

↳ all coupled   difficult to solve

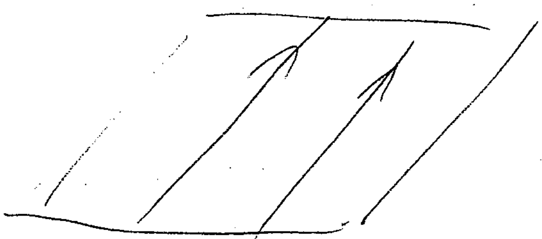
Only half of the story for mechanics

Newton's law: $\vec{F} = m\vec{a}$

According to GR: spacetime is curved

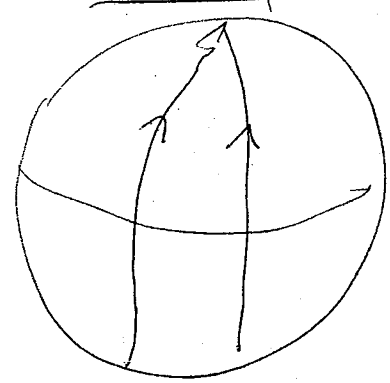
$\Rightarrow$ particles move in the straightest & shortest possible path known as Geodesic.

Flatspace



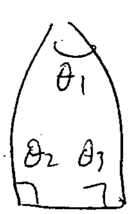
straight lines

curved space



great circles
(international flights!

Moreover, non-Euclidean

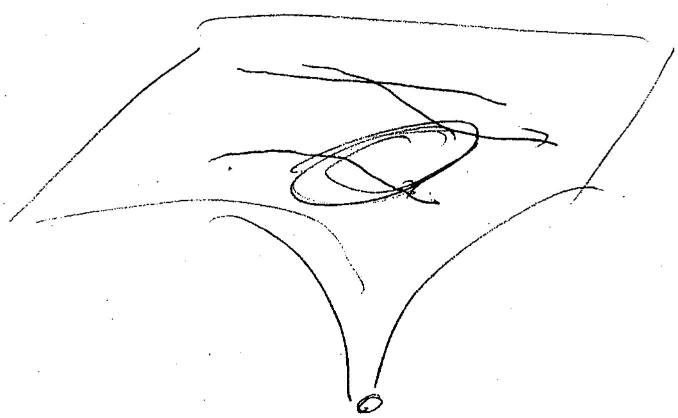


$\theta_1+\theta_2+\theta_3 > \pi$

positive curvature

Instead of the action-at-a-distance view of Newton:

: Massive objects (like, sun) warp space time



Particles follow geodesics and fall towards the massive object (e.g. sun)

Bending of light revisited

$ds^2 = 0$ light

$$\left(\frac{dr}{dt}\right)^2 = \left(1 + \frac{2\phi}{c^2}\right)^2 \simeq 1 + \frac{4\phi}{c^2}$$

as compared with previous answer simply from EP

$$\left(\frac{dr}{dt}\right)^2 = 1 + \frac{2\phi}{c^2}$$

without space warp

A factor of 2 !

$$\Rightarrow \quad \delta = 2 \times \left(\frac{2GM}{c^2 r_{min}}\right)$$

correct answer confirmed by expt

Black holes

$$ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \frac{dr^2}{1+\frac{2\Phi}{c^2}} + r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$= -\left(1-\frac{r^*}{r}\right)dt^2 + \left(1-\frac{r^*}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

where $r^* = \frac{2GM}{c^2}$ = Schwarzchild radius

An object with its mass so compressed that its radius $< r^*$ is called a black hole because it is impossible to transmit any signal (light) from $r < r^*$ region!

From an observer far away from the source pt of view, gravitational time dilation is infinite

$$dt = \frac{d\tau}{\sqrt{1-\frac{r^*}{r}}} \to \infty \quad \text{as } r \to r^*$$

Coordinate time

Alternatively $\frac{\omega_{rec}}{\omega_{em}} = \sqrt{\frac{1+\frac{2\Phi}{c^2}(r_{em})}{1+\frac{2\Phi}{c^2}(r_{rec})}} \to 0$ as $r_{em} \to r^*$

infinite time to receive next photon

Other fascinating consequences of GR :

- correct precession of mercury's perihelion
- Gravitational waves : accelerating objects emit gravitational waves
- Cosmology