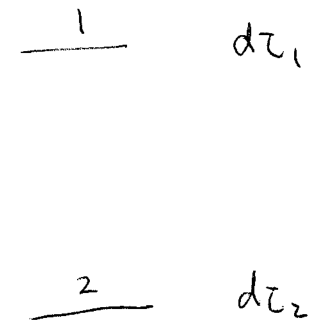
PHY 248 Lecture 5

Gary Shiu 1/31/07

How do we interpret the gravitational redshift?

Observers at emitter and receiver measure time differently.

Gravitational time dilation



$$\omega \sim \frac{1}{d\tau}$$

$$\frac{d\tau_1 - d\tau_2}{d\tau_2} = \frac{\phi_1 - \phi_2}{c^2}$$

$$\Rightarrow \quad d\tau_1 = \left(1 + \frac{\phi_1 - \phi_2}{c^2} \right) d\tau_2$$

$$= \left(\underbrace{1 + \frac{\Delta\phi}{c^2}}_{\text{time dilation}} \right) d\tau_2$$

For static gravitational field, can integrate to get

$$\tau_1 = \left(1 + \frac{\phi_1 - \phi_2}{c^2} \right) \tau_2$$

$\boxed{\text{Higher potential runs faster}}$

Note : (1) Time dilation even though there is no relative motion

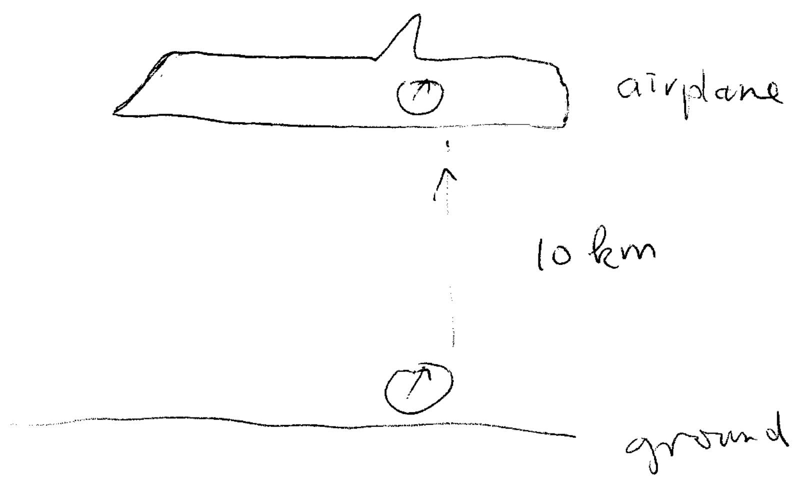
(2) Unlike time dilation in SR (i.e. each see other's clock run slower)

Here :

lower clock see upper clock run fast

upper clock " lower " " slow

(3) Time dilation by gravity has been tested

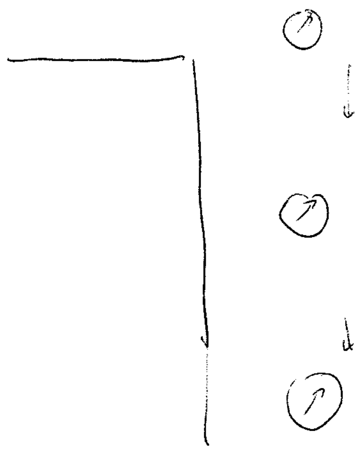


After correcting for SR effect, clock in airplane still runs fast

Also: applications in GPS (homework)

More direct derivation of time dilation

Drop a clock from a building



Go to free-falling frame (inertial frame)

$\Rightarrow$ SR applies

$$\underset{\text{free fall frame}}{t_1^{ff}} = \gamma_1 \underset{\text{rest frame}}{\tau_1} \qquad t_2^{ff} = \gamma_2 \tau_2$$

We are interested in comparing $d\tau_1$ and $d\tau_2$ where $dt_1^{ff} = dt_2^{ff}$

$$\frac{d\tau_1}{d\tau_2} = \sqrt{\frac{1 - u_1^2/c^2}{1 - u_2^2/c^2}} \simeq 1 - \frac{1}{2}\left(\frac{u_1^2 - u_2^2}{c^2}\right) + \cdots$$

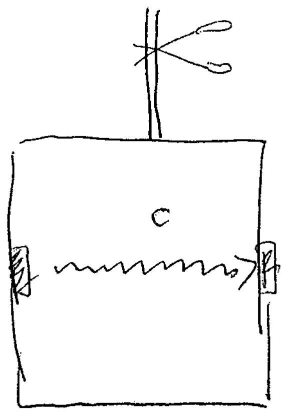
$$= 1 + \frac{\phi_1 - \phi_2}{c^2}$$

Conservation of energy

$$\frac{1}{2} m u_i^2 + \phi_i = \text{constant}$$

Bending of light

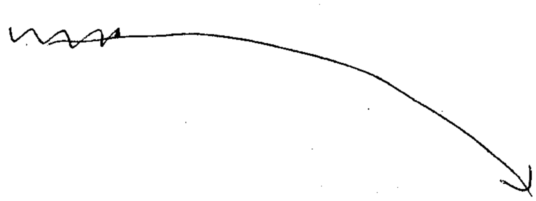
First, qualitative:



Free-falling frame = inertial frame

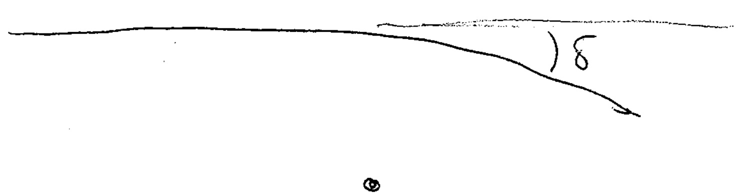
light travels horizontally with speed = c

Now according to our frame, elevator is dropping so

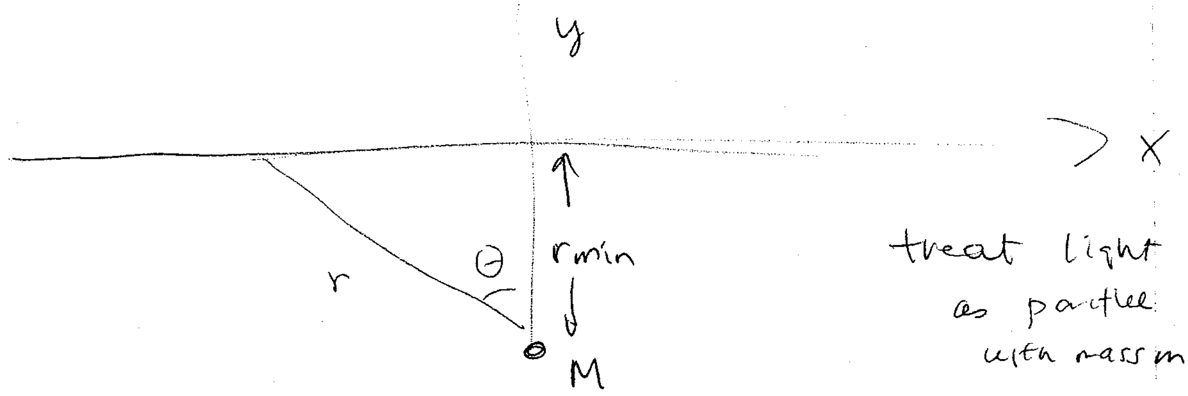


light bends like a parabola under influence of gravity

We want to calculate the angle δ light is bent in the presence of a mass M



Let us get the answer first by <u>cheating</u>



treat light as particle with mass $m = \frac{E}{c^2}$

$$\Delta p_y = \int F_y \, dt = \int F \cos\theta \, dt$$

$$= \int \frac{GMm}{r^2} \cos\theta \, dt$$

$$= \int \frac{GMm}{r^2} \left(\cos\theta\right) \frac{dx}{c}$$

$$= \int_{-\infty}^{\infty} \frac{GMm}{c} \frac{\cos\theta \, dx}{x^2 + r_{min}^2}$$

$$= \frac{GMm}{c} \cdot \int_{-\infty}^{\infty} \frac{\cos\theta \, dx}{x^2 + r_{min}^2}$$

$$= \frac{GMm}{r_{min} c} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$x = r_{min} \tan\theta$

$dx = r_{min} \sec^2\theta \, d\theta$

$$= \frac{2GMm}{r_{min} c}$$

$$\frac{\Delta p_y}{p} = \frac{2GM\overset{E/c^2}{m}}{r_{min}\, c} \, \frac{1}{E/c} = \frac{2GM}{r_{min}\, c^2}$$

Other than the fact that this is not the full story for GR, we got this result just because we were lucky.

Problems with this method

(1) Massless particle only makes sense in relativity, do not expect $\frac{GMm}{r^2}$ to work

(2) Derivation does not take into account gravitational time dilation, which is Crucial in the reasoning behind bending of light.

Bending of light (from EP)

Speed of light $\dfrac{dr}{d\tau} = c$ universal constant

According to an observer at a different position

speed appears to be different

because gravitational potential is different.

Let's choose the observer at infinity

$$t = \tau(\infty) \qquad \Phi(\infty) = 0$$

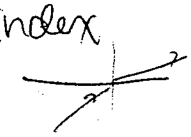
$$d\tau = \left(1 + \frac{\phi(r)}{c^2}\right) dt$$

Apparent speed of light

$$c(r) \equiv \frac{dr}{dt} = \left(1 + \frac{\phi(r)}{c^2}\right)\frac{dr}{d\tau} = \left(1 + \frac{\phi(r)}{c^2}\right)c$$

$$\equiv \frac{c}{n(r)}$$

where $n(r)$ can be interpreted as an index of refraction



$$n(r) \simeq \left(1 + \frac{\phi(r)}{c}\right)^{-1} \simeq 1 - \frac{\phi(r)}{c^2}$$

Important to note

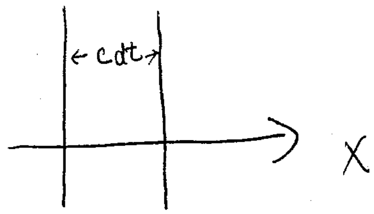
- $c = \frac{dr}{d\tau}$ is still a universal constant

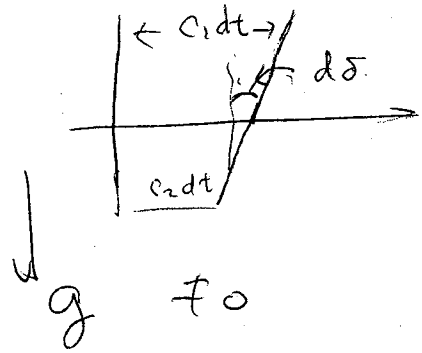
  rather it is the clock which runs differently so it appears to change according to a fixed clock

- A dramatic example : Blackhole

  Infinite time dilation $\Rightarrow$ appears to an observer at ∞ that it takes ∞ amount of time to leave the black hole even though proper time is finite.

Bending of light (the EP expectation)





$$d\delta \simeq \frac{(c_1 - c_2)\,dt}{dy} \simeq \frac{dc(r)\,\frac{dx}{c}}{dy}$$

$$\doteq \left(\frac{\partial n^{-1}}{dy}\right) dx$$

$$= -\frac{1}{n^2}\left(\frac{\partial n}{\partial y}\right) dx$$

$$\simeq -\left(\frac{\partial n}{\partial y}\right) dx$$

$$n \simeq 1 - \frac{\Phi(r)}{c^2}$$

$$\frac{\partial n}{\partial y} = \frac{\partial n}{\partial r}\frac{\partial r}{\partial y} = -\frac{1}{c^2}\left(\frac{\partial \Phi}{\partial r}\right)\frac{y}{r}$$

$$= -\frac{GM}{c^2 r^2}\frac{y}{r}$$

$$\delta = \int_{-\infty}^{\infty} \frac{GMy}{c^2 r^3}\,dx = \frac{GM}{c^2}\int_{-\infty}^{\infty} \frac{y\,dx}{r^3}$$

For small bending: $y \simeq r_{min}$

$$\Rightarrow \quad \delta \simeq \frac{GM}{c^2} \int \frac{r_{min}\, dx}{(x^2 + r_{min}^2)^{3/2}}$$

$$= \frac{2GM}{c^2 r_{min}}$$

For $r_{min} = R_\odot$ (sun) $\quad M = M_\odot$

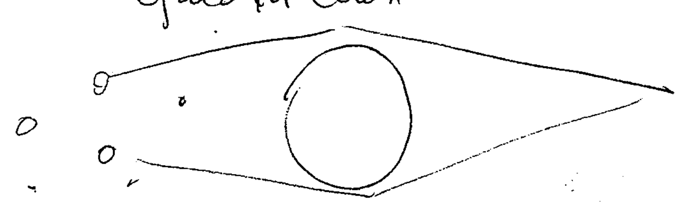
$\Rightarrow \quad \delta = 0.875''$ turns out only 1/2 of the correct answer for GR

Experimental verification of such (correct) angle of deflection is one of the greatest triumph of GR

Application

Gravitational lensing

galaxy cluster



massive object acts as a lens

see distinct galaxies that are otherwise out of sight