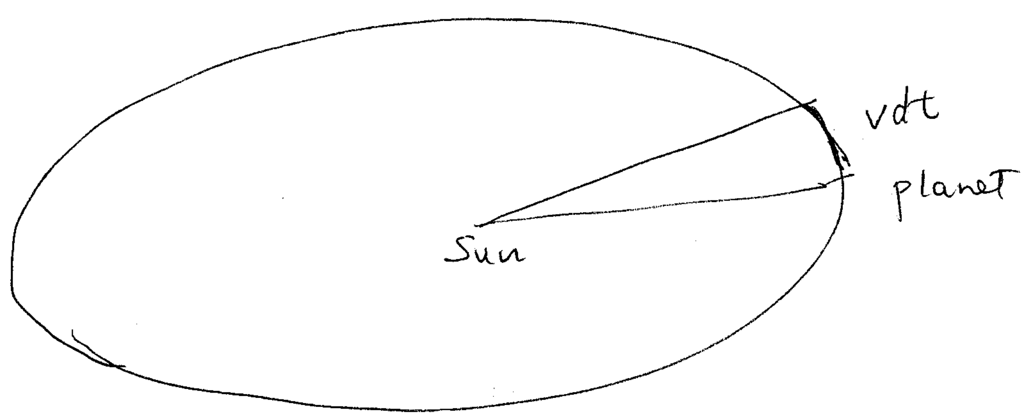
PHY 248 Lecture 3 Gary Shiu 1/26/07

Kepler's laws : A set of empirical laws for planetary motion that can be derived from Newton's laws

1st Law : planet moves in elliptical orbit with the sun at one focus

2nd Law : line joining planet to sun sweeps out equal area in equal times




$$dA = \frac{1}{2} \left| \vec{r} \times \vec{v}\, dt \right| = \frac{1}{2m} \left| \vec{r} \times \vec{p} \right| dt$$

$$\frac{dA}{dt} = \frac{1}{2m} L \qquad L = \text{angular momentum}$$

Important : $\vec{F}$ is a radial force

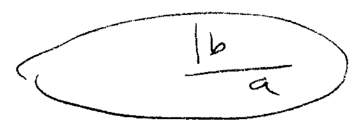
$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

$$\Rightarrow \quad L = \text{conserved}$$

$$\Rightarrow \quad \boxed{\frac{dA}{dt} = \text{constant}}$$

3rd Law : $T^2 = c\, r^3$

period      semi-major axis




a = semi-major

b = semi-minor

True for elliptical orbits in general

Let's for simplicity consider circular orbit

$$F = ma = \frac{G M_s m}{r^2}$$

Same m because of equivalence principle

$$\frac{V^2}{r} = a = \frac{G M_s}{r^2}$$

$$\Rightarrow \left(\frac{2\pi r}{T}\right)^2 = V^2 = \frac{G M_s}{r}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM_s} r^3}$$

By measuring T and r, we can weigh the Sun.

Moon does not have natural satellite, mass was not known until we introduced artificial satellites.

# Gravitational Potential Energy

Near earth's surface $F \sim \frac{GMm}{R_E^2} = mg$

$$\Rightarrow \quad U = mgh = mg(r - R_E)$$

$$\boxed{\text{chosen } U=0 \text{ at } r = R_E}$$

For distances far from earth's surface, cannot approximate $F \sim$ constant

$$dU = -\vec{F} \cdot d\vec{s}$$

$$= -\left(-\frac{GM_E m}{r^2}\right) dr$$

$$U = -\frac{GM_E m}{r} + U_o$$

constant (matter of convention)

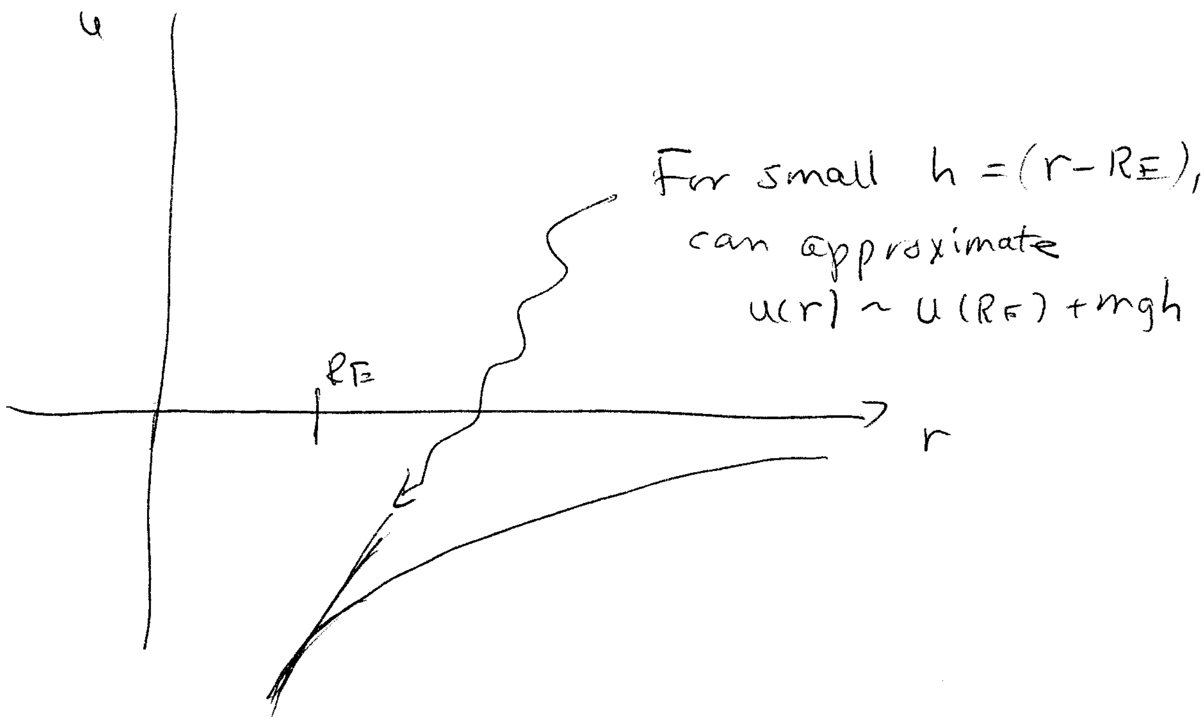
$U_o$ is not a measurable quantity, only its change (i.e. force) is.

We can choose $U_o = \frac{GM_E m}{R_E}$ such that

$$U = 0 \quad \text{on earth}$$

But a more convenient choice is $U_0 = 0$

$$\Rightarrow \quad U = -\frac{GMm}{r} \qquad \text{so } U \to 0 \text{ at infinity}$$




With this choice of $U_0$, conservation of energy reads

$$K + U = \text{constant}$$

$$\boxed{\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}}$$

## Escape Velocity

∵ gravitational pull, velocity ↓ as r ↑

Escape velocity = min vel needed for object to escape

Energy conservation

$$\frac{1}{2} m v_i^2 - \frac{G M_E m}{R_E} = \overset{\overset{0}{\shortparallel}}{K_f} + \overset{\overset{0}{\shortparallel}}{U_f} \geq 0$$

$$\Rightarrow \quad V_i \geq \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2g R_E}$$

$$\Rightarrow \quad \boxed{V_e = \sqrt{2g R_E}} \quad = 11.2 \text{ km/s} \simeq 25{,}000 \text{ mi/h}$$

Escape velocity determines the kind of atmosphere a planet can have.

Average K.E. ~ temperature (thermodynam

$$\frac{1}{2} m v^2 \sim \text{temp.}$$

⇒ heavy element ≃ small velocity

For earth, escape vel ~ thermal vel of oxygen & nitrogen molecules.

For moon $V_e$ is smaller because $M_M < M_E$

$\Rightarrow$ too small for any atmosphere to exist.

<u>Examples</u> A projectile is fired up from the surface of the earth with

(i) $v_i < v_e$, what is max height (neglect air resistance)?

(ii) $v_i > v_e$, what is velocity when it is very far away?

(i) Conservation of energy

$$0 - \frac{GMm}{r_f} = \frac{1}{2}mv_i^2 - \frac{GM_Em}{R_E^2}$$

$$\frac{1}{r_f} = -\frac{v_i^2}{2GM_E} + \frac{1}{R_E}$$

$$\boxed{r_f = \frac{1}{\frac{1}{R_E} - \frac{V_i^2}{2gR_E^2}} = \frac{R_E}{1 - \frac{V_i^2}{2gR_E}}}$$

$$h = r_f - R_E$$

(ii) Conservation of energy

$$\frac{1}{2}mv_f^2 - 0 = \frac{1}{2}mV_i^2 - \frac{GMm}{R_E^2}$$

$$v_f^2 = v_i^2 - \frac{2GM_E}{R_E} = v_i^2 - 2\frac{gR_E^2}{R_E}$$

$$\boxed{v_f = \sqrt{v_i^2 - 2gR_E}}$$

# Gravitational Field $\vec{g}$

$$\vec{F}_{1,2} = -\frac{G m_1 m_2}{r_{12}^2} \hat{r}_{1,2}$$

Force acting on $m_2$ due to $m_1$

$$\vec{F}_2 = -\sum_{i \neq 2} \frac{G m_i m_2}{r_{i2}^2} \hat{r}_{i2}$$

total force due to all masses other than $m_2$

$$\vec{g} = \frac{\vec{F}_2}{m_2}$$ = gravitational field at $\vec{r}_2$

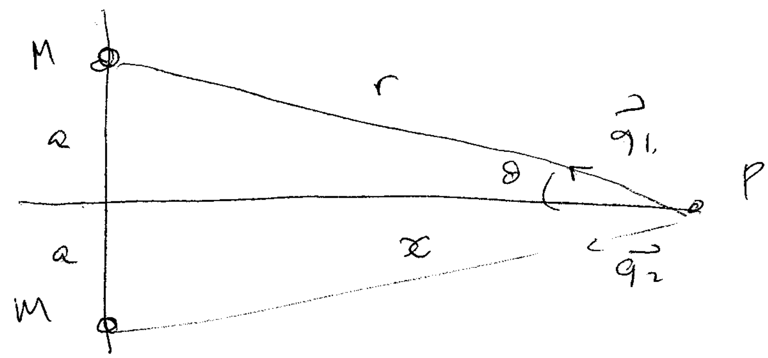
Concept of a field is very important

- not only limited to gravity but also electric & magnetic field

- Intrinsic property of a position in space, not the particular object located there

- Summarize the effect of gravity

Example : Concept seems abstract, let's look at an example




$$g_1 = \frac{GM}{r^2}$$ pointing in the direction shown

From symmetry, y component cancels.

$$g_x = g_{1x} + g_{2x} = 2g_{1x} = 2g_1 \cos\theta$$

but $\cos\theta = \frac{x}{r}$

$$\Rightarrow \vec{g} = g_x \hat{i} = -2\frac{GM}{r^2}\frac{x}{r}\hat{i}$$

$$= -\frac{2GMx}{r^3}\hat{i}$$

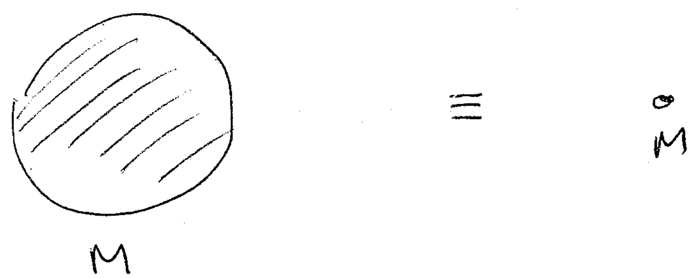
$$= -\frac{2GMx}{(x^2+a^2)^{3/2}}\hat{i}$$

Check. $x \gg a$ $\quad \vec{g} = -2\frac{GM}{x^2}\hat{i}$ total mass = 2M at $x$ away from P

## Spherical Shell and Solid Sphere

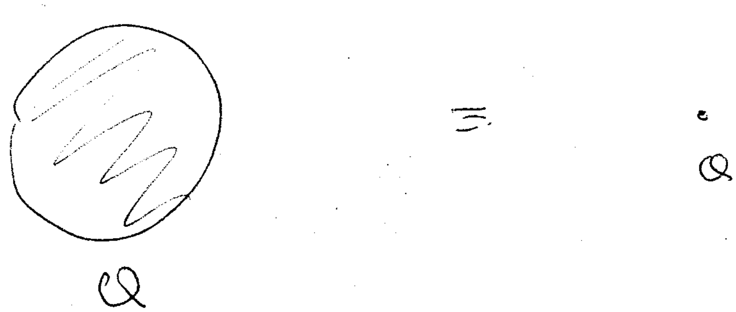
Newton's motivation for inventing calculus

show that $\vec{g}$ outside a spherically symmetric sphere is the same as if the mass is all concentrated at centre



No need to assume constant density
only $\rho(r)$

In fact $\vec{E} \sim \frac{q}{r^2}$ suggests that the same applies to electric field



We will be able to solve this problem in a split second using Gauss's law which we will discuss later when we study electromagnetism.

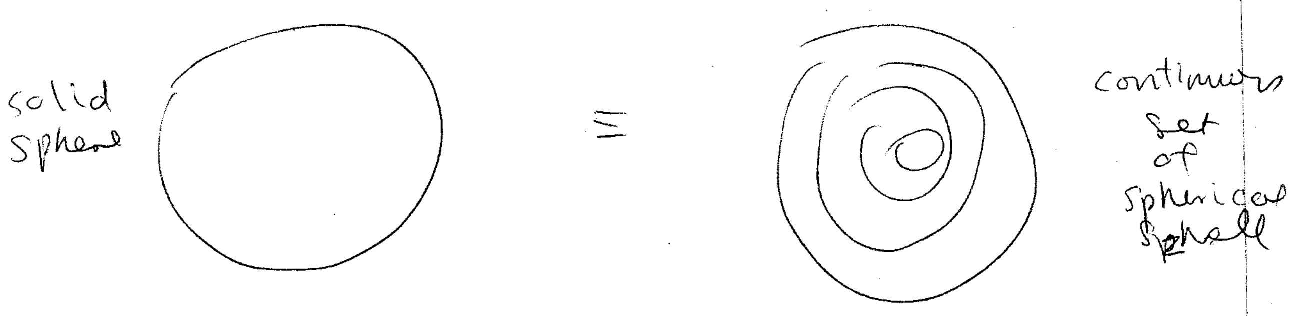
For now, do it the hard way (calculus)

Introduce Gauss's law briefly here

As a subproblem, it is easier to show that for a spherical shell

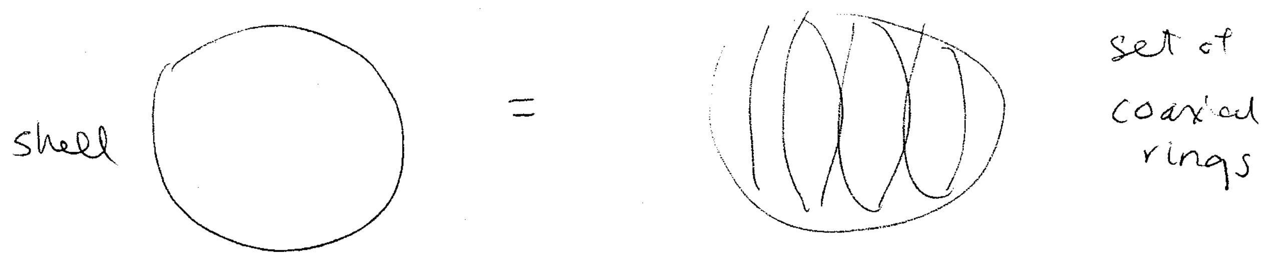
$$\vec{g} = \begin{cases} -\dfrac{GM}{r^2}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

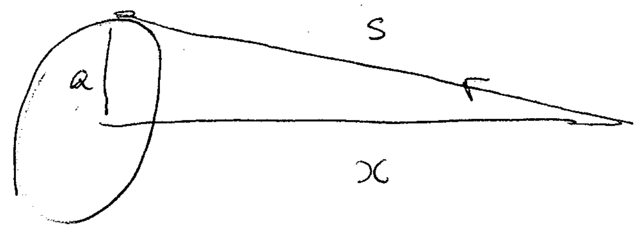
Once this is shown, the results for the solid sphere follows since we can think of




For each shell, $\vec{g}$ outside is the same as if the total mass is concentrated at center, must be the case for the solid sphere as well.

Spherical shell problem: break it down even further by considering a uniform ring




$$dg = \frac{G\,dm}{s^2}$$

From symmetry, only x-component contributes after summing over all m on the ring

$$dg_x = -dg\cos\alpha = -\frac{G\,dm}{s^2}\cos\alpha$$

$$g_x = -\int \frac{G\cos\alpha}{s^2}\,dm = -\frac{G\,m\cos\alpha}{s^2}$$

since $\alpha$ and $s$ are the same for all pts on the ring

Now, apply this result to the spherical shell problem

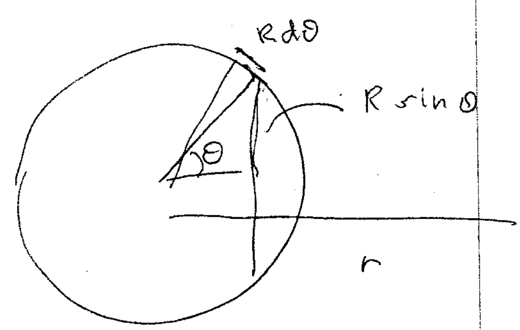
First consider a pt outside the shell.

By ~~dg~~ symmetry, the field must be radial

$$dg_r = -\frac{G\,dM}{s^2}\cos\alpha$$

where $dM = \sigma\, dA = \frac{M}{4\pi R^2}\cdot(2\pi R\sin\theta)(R\,d\theta)$

$$= \frac{1}{2} M \sin\theta\, d\theta$$




$$\Rightarrow \quad dg_r = -\frac{G\,dM}{s^2}\cos\alpha$$

$$= -\frac{GM\sin\theta\, d\theta}{2s^2}\cos\alpha$$

The variables $s$, $\theta$ and $\alpha$ are related

can rewrite integration over $\theta$ to integral over $s$

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

$$2s\,ds = +2rR\sin\theta\, d\theta$$

$$\sin\theta\, d\theta = \frac{s\,ds}{rR}$$

$$R^2 = s^2 + r^2 - 2sr\cos\alpha$$

$$\cos\alpha = \frac{s^2+r^2-R^2}{2sr}$$

Hence $$dg_r = -\frac{GM\sin\theta\, d\theta}{2s^2}\cos\alpha$$

$$= -\frac{GM}{2s^2}\left(\frac{s\,ds}{rR}\right)\left(\frac{s^2+r^2-R^2}{2sr}\right)$$

$$= -\frac{GM}{4r^2R}\left(1+\frac{r^2-R^2}{s^2}\right)ds$$

$$g_r = -\frac{GM}{4r^2R}\int_{r-R}^{r+R}\left[1+\frac{(r-R)(r+R)}{s^2}\right]ds$$

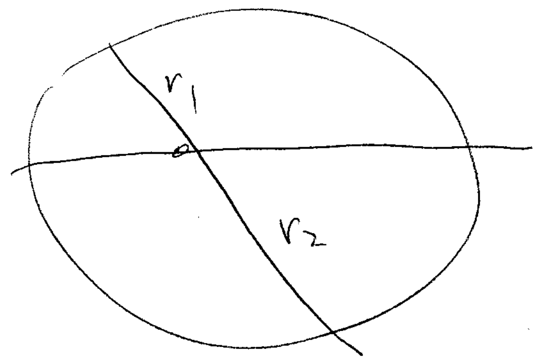
$$= -\frac{GM}{4r^2R}\left[s - \frac{(r-R)(r+R)}{s}\right]_{r-R}^{r+R}$$

$$= -\frac{GM}{r^2} \quad \text{for } r > R$$

For a pt P inside the shell, only difference is integration limit is from $R-r$ to $R+r$

$$g_r = 0$$

How do we understand that $\vec{g} = 0$ inside the shell




$$\frac{m_1}{m_2} = \frac{r_1^2}{r_2^2}$$

$$\Rightarrow \frac{m_1}{r_1^2} = \frac{m_2}{r_2^2}$$

forces cancel each other

smaller mass closer

larger mass – further