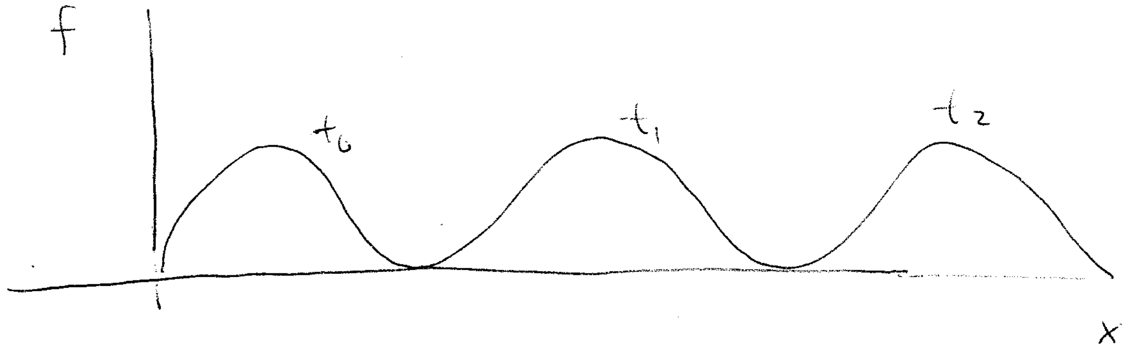
Waves (T & M ch 15, 16, 33)



$f(x-vt)$ moving in positive x direction with velocity v

$V =$ wave speed

String $\quad V = \sqrt{\dfrac{F_T}{\mu}}$ → tension; μ → mass / length

guitar strings have a wide spectrum of tension

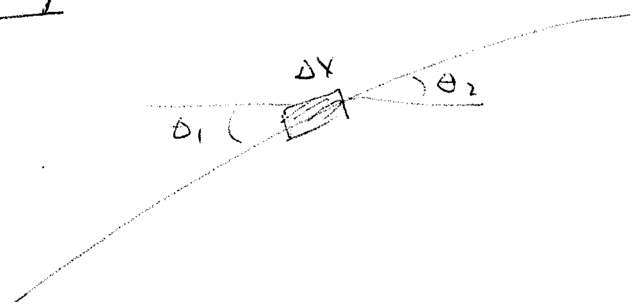
Sound $\quad V = \sqrt{\dfrac{\gamma R T}{M}}$

$\gamma =$ dependent on kind of gas
$R =$ universal gas const
$T =$ temperature
$M =$ molar mass

light $\quad V = c$

particle wave $\quad V = V$ ← velocity of particles

String



y-component of F acting on Δx

$= F_T \sin\theta_2 - F_T \sin\theta_1$

$\simeq F_T (\tan\theta_2) - F_T \tan\theta_1$

for small θ_i

$$\Rightarrow \quad F_y \approx F_T \left(\left.\frac{dy}{dx}\right|_2 - \left.\frac{dy}{dx}\right|_1 \right)$$

$$= F_T \frac{d^2y}{dx^2} \Delta x$$

But $\quad F_y = m a_y = m \dfrac{d^2y}{dt^2} = \mu \Delta x \dfrac{d^2y}{dt^2}$

$$\Rightarrow \quad F_T \frac{\partial^2 y}{\partial x^2} = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad \underbrace{\frac{F_T}{\mu}}_{V^2} \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

In general:

1-D wave equation $\quad V^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$

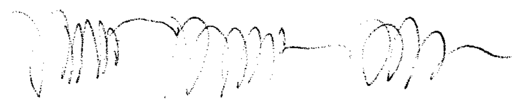
3D wave equation $\quad V^2 \left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} \right) = \dfrac{\partial^2 f}{\partial t^2}$

Let's go back to 1D case first:

Want to show $y = g(x - vt)$ is a solution

Notation : guitar string : transverse wave

slinky : longitudinal wave



If $y = g(x - vt) \equiv g(u)$ e.g. $g = \sin(x - vt)$

$$\frac{\partial y}{\partial t} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial t} = -v \frac{\partial g}{\partial u}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 g}{\partial u^2} \left(\frac{\partial u}{\partial t}\right)(-v) = v^2 \frac{\partial^2 g}{\partial u^2}$$

$$\frac{\partial y}{\partial x} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial g}{\partial u}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 g}{\partial u^2}$$

$\Rightarrow$ $y = g(x - vt)$ satisfies the wave equation

Similarly, $y = g(x + vt)$ also " " " " " " " "

Harmonic Waves

$$\sin(kx-\omega t) \quad \text{or} \quad e^{i(kx-\omega t)} \quad \text{(real part)}$$

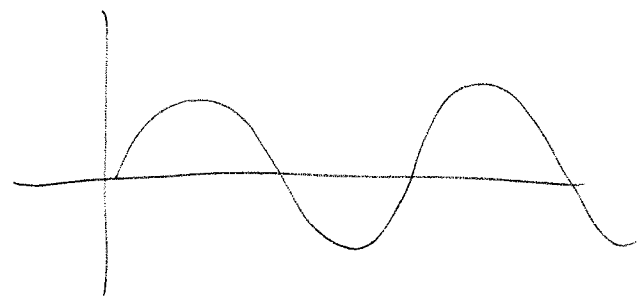
$$e^{i(kx-\omega t)} = e^{ik(x-vt)}$$

where $\frac{\omega}{k} = v$

$\Rightarrow \quad \omega = \frac{2\pi}{\lambda} \cdot v$

$\Rightarrow \quad 2\pi f = \frac{2\pi}{\lambda} v$

$\Rightarrow \quad \boxed{v = f\lambda}$



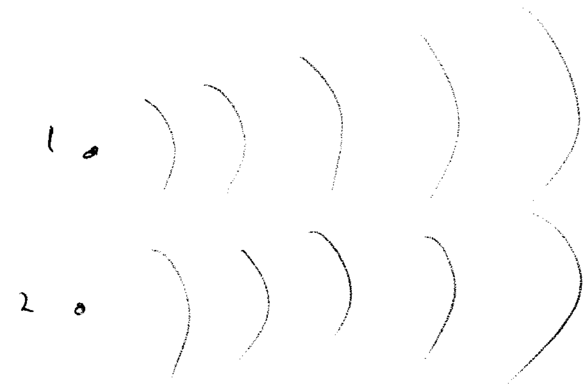
$\Delta x = \lambda$ waves repeat itself

$k\lambda = 2\pi$

$\Rightarrow \quad k = \frac{2\pi}{\lambda}$

"wave vector"

Interference



$$f = f_1 + f_2$$
$$= e^{ikr_1 - i\omega t} + e^{ikr_2 - i\omega t}$$
$$= e^{-i\omega t}\left(e^{ikr_1} + e^{ikr_2}\right)$$

$$\text{Intensity} = \frac{\text{power}}{\text{area}} \quad \propto |f|^2$$

Why? Energy transfer via wave $\propto |f|^2$

e.g. string $\quad y = A\sin(kx - \omega t)$

$$P = F_{T_y} V_y = -F_T \cancel{\tan}\theta\, V_y = -F_T \frac{dy}{dx}\frac{\partial y}{\partial t}$$

$$= -F_T\left[kA\cos(kx-\omega t)\right]\left[-A\omega\cos(kx-\omega t)\right]$$

$$= \mu v^2 \omega^2 A^2 \cos^2(kx - \omega t)$$

$$\langle P \rangle = \frac{1}{2}\mu v^2 \omega^2 A^2$$

time average of $\cos^2(kx - \omega t) = \frac{1}{2}$

Back to interference

$$\text{Intensity} \propto |f|^2 = \left| e^{-i\omega t} \left(e^{ikr_1} + e^{ikr_2} \right) \right|^2$$

$$= e^{-i\omega t} \left(e^{ikr_1} + e^{ikr_2} \right) e^{i\omega t} \left(e^{-ikr_1} + e^{-ikr_2} \right)$$

$$= 1 + 1 + e^{ik(r_1-r_2)} + e^{-ik(r_1-r_2)}$$

$$= 2 + 2\cos\left[k(r_1-r_2) \right]$$

Constructive interference

$$k(r_1-r_2) = m \cdot 2\pi$$

$$\frac{2\pi}{\lambda}(r_1-r_2) = 2\pi m$$

$$\underset{\text{integer}}{m}\lambda = r_1-r_2$$

Destructive interference

$$k(r_1-r_2) = (2m+1)\pi$$

$$\underset{\text{half integer}}{\left(m+\tfrac{1}{2}\right)}\lambda = r_1-r_2$$

For 3D Waves

$$\text{Intensity} = \frac{\text{power}}{\text{area}}$$

$$= \frac{\text{energy}}{\text{area} \cdot \text{time}}$$

$$= \frac{\text{energy}}{\text{volume}} \; \frac{\text{distance}}{\text{time}}$$

$$\propto v\,|f|^2$$ like we argued before

Conservation of energy

$$P = I \cdot \text{area}$$

$$= I \cdot 4\pi r^2$$

$$= \text{constant}$$

$$I = \frac{\text{constant}}{4\pi r^2} \propto v\,|f|^2$$

Spherical wave $\quad f \propto \frac{1}{r}$

$$f = \frac{e^{ikr - i\omega t}}{r}$$